A Method for Deriving the Dirac Equation from the Relativistic Newton's Second Law

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Abstract

The derivation becomes possible when we find a new formalism which connects the relativistic mechanics with the quantum mechanics. In this paper, we explore the quantum wave nature from the Newtonian mechanics by using a concept: velocity field. At first, we rewrite the relativistic Newton's second law as a field equation in terms of the velocity field, which directly reveals a new relationship connecting to the quantum mechanics. Next, we show that the Dirac equation can be derived from the field equation in a rigorous and consistent manner.

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In the past century, many attempts were made to address to understand the quantum wave nature from the classical mechanics, however, much of the connection with the classical physics is rather indirect. In this paper, we propose a concept: velocity field, and show that the Dirac equation may be derived from the relativistic Newton's second law in terms of the velocity field in a rigorous manner.

According to the Newtonian mechanics, in a hydrogen atom, the single electron revolves in an orbit about the nucleus, its motion can be described with its position in an inertial Cartesian coordinate System S : $(x_1, x_2, x_3, x_4 = ict)$. As the time elapses, the electron draws a spiral path (or orbit), as shown in Fig.1(a) in imagination.

If the reference frame S rotates through an angle about the x_2-axis in Fig.1(a), becomes a new reference frame S', there will be a Lorentz transformation linking the frames S and S'. Then in the frame S', the spiral path of the electron tilts with respect to the x'_4-axis with the angle as shown in Fig.1(b). At one moment, for example, $t'_4 = t_0$ moment, the spiral path pierces many points at

the plane $t'_4 = t_0$, for example, the points labeled a, b and c in Fig.1(b), these points indicate that the electron can appear at many points at the time t_0, in agreement with the concept of the probability in quantum mechanics. This situation gives us a hint for deriving quantum wave nature from the Newtonian mechanics.

Because the electron pierces the plane $t'_4 = t_0$ with 4-vector velocity u, at every pierced point we can label a local 4-vector velocity . The pierced points may be numerous if the path winds up itself into a cell about the nucleus (due to a nonlinear effect in a sense), then the 4-vector velocities at the pierced points form a 4-vector velocity field. It is noted that the observation plane selected for the piercing can be taken at an arbitrary orientation, so the 4-vector velocity field may be expressed in general as $u(x_1, x_2, x_3, x_4 = ict)$, i.e. the velocity u is of a function of position.

At every point in the reference frame S' the electron satisfies the relativistic Newton's second law

$$m\frac{du_\mu}{d\tau} = qF_{\mu\nu}u_\nu \tag{1}$$

the notations consist with the convention[1]. Since the Cartesian coordinate system is a frame of reference whose axes are orthogonal to one another, there is no distinction between covariant and contravariant components, only subscripts need be used. Here and below, summation over twice repeated indices is implied in all case, Greek indices will take on the values 1,2,3,4, and regarding the mass m as a constant. As mentioned above, the 4-vector velocity u can be regarded as a 4-vector velocity field, then

$$\frac{du_\mu}{d\tau} = \frac{\partial u_\mu}{\partial x_\nu}\frac{dx_\nu}{d\tau} = u_\nu \partial_\nu u_\mu \tag{2}$$

$$qF_{\mu\nu}u_\nu = qu_\nu(\partial_\mu A_\nu - \partial_\nu A_\mu) \tag{3}$$

Substituting them back into Eq.(1), and re-arranging these terms, we obtain
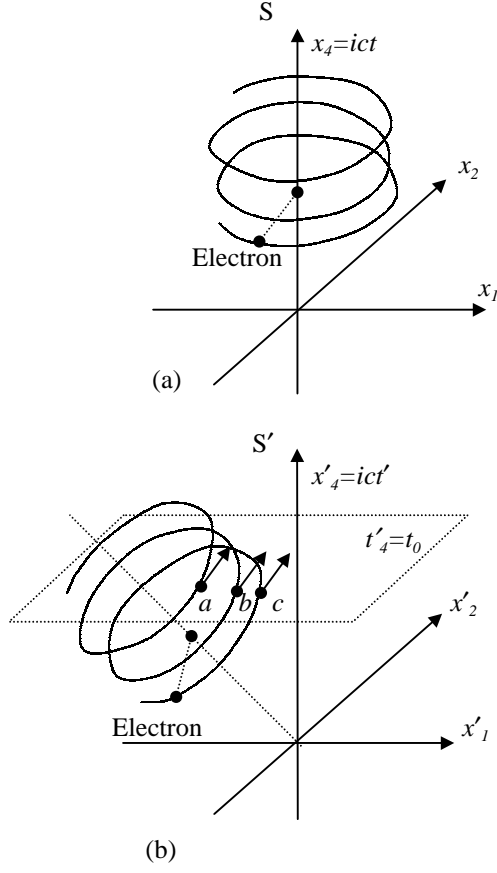
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Figure 1: The motion of the electron of hydrogen atom in 4-dimensional space-time.

$$u_\nu \partial_\nu(mu_\mu + qA_\mu) = u_\nu \partial_\mu(qA_\nu)$$
$$= u_\nu \partial_\mu(mu_\nu + qA_\nu) - u_\nu \partial_\mu(mu_\nu)$$
$$= u_\nu \partial_\mu(mu_\nu + qA_\nu) - \frac{1}{2}\partial_\mu(mu_\nu u_\nu)$$
$$= u_\nu \partial_\mu(mu_\nu + qA_\nu) - \frac{1}{2}\partial_\mu(-mc^2)$$
$$= u_\nu \partial_\mu(mu_\nu + qA_\nu) \qquad (4)$$

Using the notation

$$K_{\mu\nu} = \partial_\mu(mu_\nu + qA_\nu) - \partial_\nu(mu_\mu + qA_\mu) \qquad (5)$$

Eq.(4) is given by

$$u_\nu K_{\mu\nu} = 0 \qquad (6)$$

Because $K_{\mu\nu}$ contains the variables $\partial_\mu u_\nu$, $\partial_\mu A_\nu$, $\partial_\nu u_\mu$ and $\partial_\nu A_\mu$ which are independent from u_ν, then a solution satisfying Eq.(6) is of

$$K_{\mu\nu} = 0 \qquad (7)$$
$$\partial_\mu(mu_\nu + qA_\nu) = \partial_\nu(mu_\mu + qA_\mu) \qquad (8)$$

The above equation allows us to introduce a potential function Φ in mathematics, further set $\Phi = -i\hbar \ln \psi$, we obtain a very important equation

$$(mu_\mu + qA_\mu)\psi = -i\hbar \partial_\mu \psi \qquad (9)$$

where ψ representing the wave nature may be a complex mathematical function, its physical meanings will be determined from experiments after the introduction of the Planck's constant $\hbar$.

Multiplying the two sides of the following familiar equation by ψ

$$-m^2 c^2 = m^2 u_\mu u_\mu \qquad (10)$$

which is valid at every point in the 4-vector velocity field, and using Eq.(9), we obtain

$$-m^2 c^2 \psi = mu_\mu(-i\hbar \partial_\mu - qA_\mu)\psi$$
$$= (-i\hbar \partial_\mu - qA_\mu)(mu_\mu \psi) - [-i\hbar \psi \partial_\mu(mu_\mu)]$$
$$= (-i\hbar \partial_\mu - qA_\mu)(-i\hbar \partial_\mu - qA_\mu)\psi$$
$$\quad - [-i\hbar \psi \partial_\mu(mu_\mu)] \qquad (11)$$

According to the continuity condition for the electron motion

$$\partial_\mu(mu_\mu) = 0 \qquad (12)$$

we have

$$-m^2 c^2 \psi = (-i\hbar \partial_\mu - qA_\mu)(-i\hbar \partial_\mu - qA_\mu)\psi \qquad (13)$$

It is known as the Klein-Gordon equation.

On the condition of non-relativity, the Schrodinger equation can be derived from the Klein-Gordon equation [3](P.469).

However, we must admit that we are careless when we use the continuity condition Eq.(12), because, from Eq.(9) we obtain

$$\partial_\mu(mu_\mu) = \partial_\mu(-i\hbar \partial_\mu \ln \psi - qA_\mu) = -i\hbar \partial_\mu \partial_\mu \ln \psi \qquad (14)$$

where we have used the Lorentz gauge condition. Thus from Eq.(10) to Eq.(11) we obtain

$$-m^2 c^2 \psi = (-i\hbar \partial_\mu - qA_\mu)(-i\hbar \partial_\mu - qA_\mu)\psi$$
$$\quad + \hbar^2 \psi \partial_\mu \partial_\mu \ln \psi \qquad (15)$$

This is of a complete wave equation for describing accurately the motion of the electron. The Klein-Gordon

equation is a linear equation so that the principle of superposition remains valid, however with the addition of the last term of Eq.(15), Eq.(15) turns to display chaos.

In the following we shall show the Dirac equation from Eq.(9) and Eq.(10). From Eq.(9), the wave function can be given in integral form by

$$\Phi = -i\hbar \ln \psi = \int_{x_0}^{x} (mu_\mu + qA_\mu)dx_\mu + \theta \qquad (16)$$

where θ is an integral constant, x_0 and x are the initial and final points of the integral with an arbitrary integral path. Since the Maxwell's equations are gauge invariant, Eq.(9) should preserve invariant form under a gauge transformation specified by

$$A'_\mu = A_\mu + \partial_\mu \chi, \quad \psi' \leftarrow \psi \qquad (17)$$

where χ is an arbitrary function. Then Eq.(16) under the gauge transformation is given by

$$
\begin{aligned}
\psi' &= \exp\left\{ \frac{i}{\hbar} \int_{x_0}^{x} (mu_\mu + qA_\mu)dx_\mu + \frac{i}{\hbar}\theta \right\} \exp\left\{ \frac{i}{\hbar}q\chi \right\} \\
&= \psi \exp\left\{ \frac{i}{\hbar}q\chi \right\} \qquad (18)
\end{aligned}
$$

The situation in which a wave function can be changed in a certain way without leading to any observable effects is precisely what is entailed by a symmetry or invariant principle in quantum mechanics. Here we emphasize that the invariance of velocity field is hold for the gauge transformation.

Suppose there is a family of wave functions $\psi^{(j)}, j = 1, 2, 3, ..., N$, which correspond to the same velocity field denoted by $P_\mu = mu_\mu$, they are distinguishable from their different phase angles θ as in Eq.(16). Then Eq.(10) can be given by

$$0 = P_\mu P_\mu \psi^{(j)}\psi^{(j)} + m^2c^2\psi^{(j)}\psi^{(j)} \qquad (19)$$

Suppose there are matrices a_μ which satisfy

$$a_{\nu l j}a_{\mu jk} + a_{\mu l j}a_{\nu jk} = 2\delta_{\mu\nu}\delta_{lk} \qquad (20)$$

then Eq.(19) can be rewritten as

$$
\begin{aligned}
0 &= a_{\mu kj}a_{\mu jk}P_\mu\psi^{(k)}P_\mu\psi^{(k)} \\
&+ (a_{\nu l j}a_{\mu jk} + a_{\mu l j}a_{\nu jk})P_\nu\psi^{(l)}P_\mu\psi^{(k)}|_{\nu\geq\mu,when\nu=\mu,l\neq k} \\
&+ mc\psi^{(j)}mc\psi^{(j)} \\
&= [a_{\nu l j}P_\nu\psi^{(l)} + i\delta_{lj}mc\psi^{(l)}][a_{\mu jk}P_\mu\psi^{(k)} - i\delta_{jk}mc\psi^{(k)}] \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (21)
\end{aligned}
$$

Where δ_{jk} is the Kronecker delta function, $j, k, l = 1, 2, ..., N$. For the above equation there is a special solution given by

$$[a_{\mu jk}P_\mu - i\delta_{jk}mc]\psi^{(k)} = 0 \qquad (22)$$

There are many solutions for a_μ which satisfy Eq.(20), we select a set of a_μ as

$$
\begin{aligned}
N &= 4, \quad a_\mu = \gamma_\mu \quad (\mu = 1, 2, 3, 4) \qquad (23) \\
\gamma_n &= -i\beta\alpha_n \quad (n = 1, 2, 3), \quad \gamma_4 = \beta \qquad (24)
\end{aligned}
$$

where γ_μ, α and β are the matrices defined in the Dirac algebra[1](P.557). Substituting them into Eq.(22), we obtain

$$[ic(-i\hbar\partial_4 - qA_4) + c\alpha_n(-i\hbar\partial_n - qA_n) + \beta mc^2]\psi = 0 \quad (25)$$

where ψ is an one-column matrix about $\psi^{(k)}$.

Let index s denote velocity field, then $\psi_s(x)$ whose four component functions correspond to the same velocity field s may be regarded as the eigenfunction of the velocity field s, it may be different from the eigenfunction of energy. Because the velocity field is an observable in a physical system, in quantum mechanics we know that $\psi_s(x)$ constitute a complete basis in which arbitrary function $\phi(x)$ can be expanded in terms of them

$$\phi(x) = \int C(s)\psi_s(x)ds \qquad (26)$$

Obviously, $\phi(x)$ satisfies Eq.(25). Eq.(25) is well known as the Dirac equation.

Alternatively, another method to show the Dirac equation is more traditional: At first, we show the Dirac equation of free particle by employing plane waves, we easily obtain Eq.(25) on the condition of $A_\mu = 0$; Next, adding electromagnetic field, plane waves are valid in any finite small volume with the momentum of Eq.(9) when we regard the field to be uniform in the volume, so the Dirac equation Eq.(25) is valid in the volume even if $A_\mu \neq 0$, plane waves constitute a complete basis in the volume; Third, the finite small volume can be chosen to locate at anywhere, then anywhere have the same complete basis, therefore the Dirac equation Eq.(25) is valid at anywhere.

Of course, on the condition of non-relativity, the Schrodinger equation can be derived from the Dirac equation [3](P.479).

By further calculation, The Dirac equation can arrive at the Klein-Gordon equation with an additional term which represents the effect of spin, this term is just the last term in Eq.(14) in a sense.

But, do not forget that the Dirac equation is a special solution of Eq.(21), therefore we believe there are some quantum effects beyond the Dirac equation.

We do not know exactly what kind of the path of the electron in a hydrogen atom is, so the illustration of Fig.1 is an imaginary one for visualizing the motion of the electron. But we know that the electron path will pierce

many points at any observation time plane like $t'_4 = t_0$ for arbitrary reference frame S' if the path or orbit exists in the 4-dimensional space-time, the points may be numerous. Therefore there is a 4-vector velocity field for the motion of the electron. The 4-vector velocity field is a key concept for our deduction.

It follows from Eq.(9) that the path of particle is analogous to "lines of electric force" in 4-dimensional space-time. In the case that the Klein-Gordon equation is valid, i.e. Eq.(12) is valid, at any point, the path can have but one direction (i.e. the local 4-vector velocity direction), hence only one path can pass through each point of the space-time. In other words, the path never intersects itself when it winds up itself into a cell about the nucleus. No path originates or terminates in the space-time. But, in general, the divergence of the 4-vector velocity field does not equal to zero, as indicated in Eq.(14).

In the Minkowski's space, every particle has its 4-vector velocity with the same magnitude $|u| = ic$. We now consider a situation where the electron revolves about the nucleus in the 2D plane $x_1 - x_4$ as shown in Fig.2. The first question: how the electron surpasses the nucleus when both the electron and nucleus have their own 4-vector velocities with the same magnitude $|u| = ic$? According to the theorem of relativistic addition of velocities, when the electron revolves the nucleus at the light speed with respect to the nucleus, no matter what speed of the nucleus is, the electron speed seem still being at the light speed in laboratory reference system. The second question: the electron will intersects itself path in the plane, for example, point P in Fig.2, how to explain ? Certainly, because of the intersection, the electron collides with itself, there is the divergence of 4-vector velocity field at every point, as indicated by Eq.(14). This new character profoundly accounts for the quantum wave natures such as spin effect[6].

In conclusion, the path of the electron of hydrogen atom should wind up itself into a cell about the nucleus in 4-dimensional space-time, therefore there is a 4-vector velocity field for describing the motion of the electron. In terms of the 4-vector velocity field, the relativistic Newton's second law can be rewritten as a wave field equation. By this, the Klein-Gordon equation, Schrodinger equation and Dirac equation can be derived from the relativistic Newtonian mechanics on different conditions, respectively.

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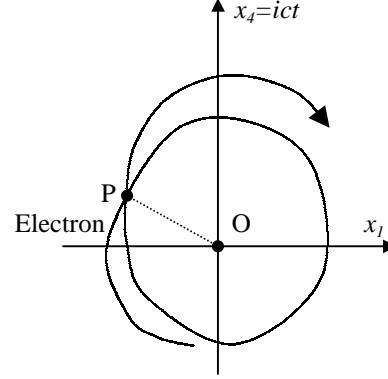[2] J. Y. Zeng, Quantum Mechanics, (Science, China, 1981).

Figure 2: The electron revolves about the nucleus in the plane $x_1 - x_4$

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